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Non-Classical Quadrature Schemes for the Approximation of Cauchy Type Oscillatory and Singular Integrals in Complex Plane

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Abstract

In this paper, non-classical numerical schemes are proposed for the approximation of Cauchy type oscillatory and strongly singular integrals in complex plane. The schemes are developed by incorporating classical quadrature rule meant for the Cauchy type complex singular integrals over a line segment in complex plane with a quasi exact quadrature method meant for the numerical integration of complex definite integrals with an oscillatory weight function. The error bounds are established and the schemes are numerically validated using a set of standard test integrals.Numerical results show that these schemes are efficient.

Keywords: analytic function; asymptotic error estimate; Cauchy principal value; error bound; Hardamad finite part integral.

1 Introduction

In this paper, we consider the numerical approximations of the following three singular integrals

$$I(u) = P\int_{\lambda-l}^{\lambda+l} \frac{u(z)}{z-\lambda}dz, \tag{1}$$

$$J(u) = P\int_{\lambda-l}^{\lambda+l} \frac{u(z)}{z-\lambda}e^{iwz}dz; w\in\mathbb{R}-\{0\}; i=\sqrt{-1}, \tag{2}$$

and

$$H(u) = H\int_{\lambda-l}^{\lambda+l} \frac{u(z)}{(z-\lambda)^n}dz; n\in\mathbb{N}-\{1\}, \tag{3}$$

where $u(z)$ is assumed to be analytic in a domain

$$\Omega = \{z\in\mathbb{C} : |z-\lambda| < \rho = r|l|, r>1\},$$

lying in the complex plane $\mathbb{C}$ containing the directed line segment L joining the points from $\lambda-l$ to $\lambda+l$. Das and Hota [5] recently have framed a derivative free one parameter interpolatory type of quadrature rule for the numerical evaluation of the integral (1.1). The rule constructed by them is claimed as the rule which numerically integrates the complex CPV integral more precisely than the rules existing in literature. Further, Bej, Hota and Das([8]) have derived some more rules of algebraic degree of precision eight involving less number of nodes (six nodes) from the rules due to Das and Hota[5] for the numerical integration of the same integral. To approximate the oscillatory integrals few works have also been done by the eminent researchers. The authors in [9] evaluated oscillatory integrals by analytic continuation. Later in 2000 generalized quadrature rules for oscillatory integrals were given in [2] whereas author of [10] proposed a new algorithm for computing Cauchy principal value integrals of oscillatory functions. However, in this paper we have constructed a family of two parameter rules of precision at least eight for the numerical approximations of complex CPV integrals of the type (1.1). Subsequently, three rules of precision ten have been constructed from the family of rules of precision at least eight and applied for the approximate evaluation of integrals of the kind (1.2) and (1.3).

2 Formulation of Rules

To construct the proposed family of rules, nodes that we have chosen are

$$\lambda \pm \alpha l \lambda \pm i\beta l,$$

along with five more nodes

$$\lambda \pm l, \lambda, \lambda \pm il,$$

due to Birkhoff-Young [1] for the numerical integration of complex line integrals. With these nodes the family of two parameter rules of precession at least eight is defined and denoted by

$$\begin{aligned} R_u(\alpha,\beta) &= w_0u(\lambda) + w_1[u(\lambda+l)-u(\lambda-l)] + w_2[u(\lambda+il)-u(\lambda-il)] + w_3[u(\lambda+\alpha l)-u(\lambda-\alpha l)] \\ &+ w_4[u(\lambda+i\beta l)-u(\lambda-i\beta l)];\ \ i=\sqrt{(-1)}; \qquad 0<\alpha\neq\beta\leq 1, \end{aligned} \tag{4}$$

for the approximation of the CPV integral given in (1.1).

Since the quadrature rule $R_u(\alpha,\beta)$ is fully symmetric and exactly integrates all even degree monomials therefore, to determine the weights w_0, w_1, w_2, w_3, w_4 and the parameters α and β we make here the assumption that

$$I((z-\lambda)^k) = R_u(\alpha,\beta)((z-\lambda)^k), \tag{5}$$

for $k = 0, 1, 3, 5$ and 7.

At this stage, the equation (5) gives

$$\begin{cases} w_0 = 0; \\ w_1 + iw_2 + \alpha w_3 + i\beta w_4 = 1; \\ w_1 - iw_2 + \alpha^3 w_3 - i\beta^3 w_4 = \frac{1}{3}; \\ w_1 + iw_2 + \alpha^5 w_3 + i\beta^5 w_4 = \frac{1}{5}; \\ w_1 - iw_2 + \alpha^7 w_3 - i\beta^7 w_4 = \frac{1}{7}; \end{cases} \tag{6}$$

and then solving it, we obtain

$$\begin{cases} w_0 = 0; \\ w_1 = \frac{18+28\beta^2-28\alpha^2-70\alpha^2\beta^2}{105(1-\alpha^2)(1+\beta^2)}; \\ w_2 = i\left[\frac{35\alpha^2\beta^2+7\alpha^2-7\beta^2-3}{105(1-\alpha^2)(1+\beta^2)}\right]; \\ w_3 = \frac{20+84\beta^2}{105(\alpha^2+\beta^2)(\alpha-\alpha^5)}; \\ w_4 = i\left[\frac{20-84\alpha^2}{105(\alpha^2+\beta^2)(\beta-\beta^5)}\right]. \end{cases} \tag{7}$$

With these weights we claim here that the rule $R_u(\alpha,\beta)$ is the quadrature rule of precision at least eight meant for the approximations of the integrals of the type (1.1). Further, to obtain the quadrature rules of precision ten we assume here that the function u is sufficiently differentiable in the disc Ω. Now by using Taylor's theorem and then, with subsequent simplifications we get

$$\begin{aligned} R_u(\alpha,\beta) &= 2lu'(\lambda) + \frac{2l^3}{3(3!)}u^{(3)}(\lambda) + \frac{2l^5}{5(5!)}u^{(5)}(\lambda) \\ &+ \frac{2l^7}{7(7!)}u^{(7)}(\lambda) + \frac{2l^9}{(9!)}\left[\frac{21+20\beta^2-20\alpha^2-84\alpha^2\beta^2}{105}\right]u^{(9)}(\lambda) \\ &+ \frac{2l^{11}}{(11!)}\left[\frac{15-20(\alpha^4+\beta^4-\alpha^2\beta^2)+84\alpha^2\beta^2(\beta^2-\alpha^2)}{105}\right]u^{(11)}(\lambda) + ... \end{aligned} \tag{8}$$

Again, denoting $E_u(\alpha,\beta)$ as a truncation error associated with the rule $R_u(\alpha,\beta)$ we get

$$I(u) = R_u(\alpha,\beta) + E_u(\alpha,\beta);$$

where

$$\begin{aligned} E_u(\alpha,\beta) &= \frac{2l^9}{(9!)}\left[\frac{252\alpha^2\beta^2+60\alpha^2-60\beta^2-28}{315}\right]u^{(9)}(\lambda) \\ &+ \frac{2l^{11}}{11!}\left[\frac{220(\alpha^4+\beta^4-\alpha^2\beta^2)-924\alpha^2\beta^2(\beta^2-\alpha^2)-60}{1155}\right]u^{(11)}(\lambda)+\dots. \end{aligned} \tag{9}$$

Now, choosing β as $\frac{1}{\sqrt{21}}$ and making the leading coefficient of $u^{(9)}(\lambda)$ as zero, $\alpha=\sqrt{\frac{3}{7}}$ is obtained. With this $(\alpha,\beta)=(\sqrt{\frac{3}{7}},\frac{1}{\sqrt{21}})$, the rule $R_u(\alpha,\beta)$ given in equation (4) reduces to a rule

$$\begin{aligned} T_1(u) &= w_0u(\lambda)+w_1[u(\lambda+l)-u(\lambda-l)]+w_2[u(\lambda+il)-u(\lambda-il)] \\ &+ w_3[u(\lambda+\alpha l)-u(\lambda-\alpha l)]+w_4[u(\lambda+i\beta l)-u(\lambda-i\beta l)], \end{aligned} \tag{10}$$

of precision ten associated with the weights

$$w_0=0;\quad w_1=\frac{31}{330};\quad w_2=i\frac{1}{375};\quad w_3=\frac{147}{250}\sqrt{\frac{7}{3}};\quad w_4=-i\frac{441\sqrt{21}}{1375}, \tag{11}$$

designed for the approximation of the CPV integral (1.1). Proceeding in the same way we obtain two more rules of precision ten

$$\begin{aligned} T_2(u) &= \frac{1501}{16170}[u(\lambda+l)-u(\lambda-l)]+i\frac{47}{19110}[u(\lambda+il)-u(\lambda-il)] \\ &+ \frac{257049}{447370}\sqrt{\frac{39}{17}}[u(\lambda+\alpha l)-u(\lambda-\alpha l)]-i\frac{177147\sqrt{27}}{528710}[u(\lambda+i\beta l)-u(\lambda-i\beta l)], \end{aligned} \tag{12}$$

and

$$\begin{aligned} T_3(u) &= \frac{997}{11040}[u(\lambda+l)-u(\lambda-l)]+i\frac{58}{28365}[u(\lambda+il)-u(\lambda-il)] \\ &+ \frac{225792}{413885}\sqrt{\frac{42}{19}}[u(\lambda+\alpha l)-u(\lambda-\alpha l)]-i\frac{107163\sqrt{63}}{292640}[u(\lambda+i\beta l)-u(\lambda-i\beta l)], \end{aligned} \tag{13}$$

for different values of $(\alpha,\beta)=\left(\sqrt{\frac{17}{39}},\frac{1}{\sqrt{27}}\right)$ and $\left(\sqrt{\frac{19}{42}},\frac{1}{\sqrt{63}}\right)$ for the approximate evaluation of the CPV integral (1.1).

Denoting $E_{T_k}(u)$ as the truncation error associated with the rules $T_k(u)$ for $k=1,2,3$ we obtain

$$\begin{aligned} E_{T_1} &= -\frac{2l^{11}}{11!}\left(\frac{7232}{509355}\right)u^{(11)}(\lambda)-\frac{2l^{13}}{13!}\left(\frac{227456}{12641265}\right)u^{(13)}(\lambda)+\dots \\ E_{T_2} &= -\frac{2l^{11}}{11!}\left(\frac{1088}{81081}\right)u^{(11)}(\lambda)-\frac{2l^{13}}{13!}\left(\frac{32128}{1848015}\right)u^{(13)}(\lambda)+\dots \\ E_{T_3} &= -\frac{2l^{11}}{11!}\left(\frac{400}{33957}\right)u^{(11)}(\lambda)-\frac{2l^{13}}{13!}\left(\frac{607384}{37923795}\right)u^{(13)}(\lambda)+\dots \end{aligned} \tag{14}$$

From the equation(2.14) it is evident that all the rules $T_k(u)$; for $k = 1, 2$ and 3 are the rules of precision ten and are used to approximate the integral (1.1) numerically.

3 Scheme for the Approximation of $J(u)$

$$J(u) = P\int_{\lambda-l}^{\lambda+l} \frac{u(z)}{z-\lambda} \cos wz dz; w \in \mathbb{R} - \{0\}.$$

Though, the integral

$$\begin{aligned} J(u) &= \int_L e^{iwz} \frac{u(z)}{z-\lambda} dz, \\ &= \int_L \frac{g(z)}{z-\lambda} dz; g(z) = e^{iwz} u(z), \end{aligned}$$

is an integral of Cauchy type and belongs to the class of integrals which can be numerically evaluated by the family of rules as proposed in the above section, but its degree of accuracy decreases rapidly as $|w|$ increases from $|w| = 1(1)5$ and finally diverges significantly for large value of $|w| \geq 6$; when the same rule is applied for its numerical approximation. It is pertinent to note here that with the increasing value of $|w|$ the oscillations of e^{iwz} causes the oscillation of the integrand function and thus, a severe cancellation occurs in the process of numerical approximation.

Looking to these limitations, we have devised here an efficient numerical scheme with the help of quadrature rule constructed for the numerical evaluation of complex CPV integrals of the type(1.1) and a quasi-exact non-classical quadrature rule analogous to Filon type rules for the approximate evaluation of the Cauchy type oscillatory integral

$$J(u) = P\int_{\lambda-l}^{\lambda+l} e^{iwz} \frac{u(z)}{z-\lambda} dz;$$

where $u(z)$ is a smooth function. Further, the scheme also has been validated numerically by some common integrals.

To construct the scheme, we rewrite the integral $J(u)$ as

$$\begin{aligned} J(u) &= \int_{\lambda-l}^{\lambda+l} e^{iwz} \frac{u(z)}{z-\lambda} dz \\ &= \int_{\lambda-l}^{\lambda+l} (e^{iwz} - e^{iw\lambda}) \frac{u(z)}{z-\lambda} dz + e^{iw\lambda} \int_{\lambda-l}^{\lambda+l} \frac{u(z)}{z-\lambda} dz \\ &= J_o(u) + e^{iw\lambda} I(u), \end{aligned} \tag{15}$$

where

$$J_o(u) = \int_{\lambda-l}^{\lambda+l} (e^{iwz} - e^{iw\lambda}) \frac{u(z)}{z-\lambda} dz,$$

is an oscillatory integral without having any Cauchy type singularities and

$$I(u) = \int_{\lambda-l}^{\lambda+l} \frac{u(z)}{z-\lambda} dz,$$

is a singular integral of type (1.1) and can be approximated numerically by the qudrature rules as projected in the above section.

Furthermore, because a large value of $|w|$ generates strong oscillations of the integrand function of the first integral $J_o(u)$,the outcome of its frequently has a negative impact on the result of the second singular integral's $(I(u))$ approximation, and therefore a desired precision for the integral $J(u)$ may not be obtained. Consequently, to get around this difficulty, we suggest a quickly convergent quasi-exact approach that nearly perfectly integrates the integral $J_o(u)$.

3.1 The Proposed Quasi-Exact Method

Let the function $u(z)$ is continuous and infinitely differentiable in the complex plane $\mathbb{C}$. Then, expanding $u(z)$ by using Taylor's expansion about the point of singularity $z = \lambda$ we get

$$u(z) = \sum_{k=0}^{\infty} c_k (z-\lambda)^k,$$

where $c_k = \frac{u^{(k)}(\lambda)}{k!}$ are the Taylor's coefficients. Truncating the above series after the first $(n+1)$ terms, the interpolating polynomial $g_n(x)$ with the interpolating condition

$$g_n^{(i)}(z) = u^{(i)}(z); \forall i = 0(1)n,$$

is obtained as

$$g_n(z) = u(\lambda) + \sum_{k=1}^{n} \frac{u^{(k)}(\lambda)}{k!} (z-\lambda)^k.$$

Using the standard method[8] it can be proved that the truncation error $\widetilde{E}_n(u)$ associated with the polynomial $g_n(z)$ is

$$\widetilde{E}_n(u) = \frac{(z-\lambda)^{n+1}}{(n+1)!} u^{(n+1)}(\xi),$$

for ξ an arbitrary point on the line segment L joining the point $\lambda - l$ to $\lambda + l$. Now,

$$u(z) \simeq g_n(z).$$

Thus,

$$\begin{aligned} J_o(u) &= \int_{\lambda-l}^{\lambda+l}(e^{iwz}-e^{iw\lambda})\frac{u(z)}{z-\lambda}dz \\ &\simeq \int_{\lambda-l}^{\lambda+l}(e^{iwz}-e^{iw\lambda})\frac{g_n(z)}{z-\lambda}dz \\ &= u(\lambda)\int_{\lambda-l}^{\lambda+l}\frac{e^{iwz}-e^{iw\lambda}}{z-\lambda}dz+\sum_{k=1}^{\infty}\frac{u^{(k)}(\lambda)}{k!}\int_{\lambda-l}^{\lambda+l}(z-\lambda)^{k-1}(e^{iwz}-e^{iw\lambda})dz \\ &= u(\lambda)(J_c+iJ_s)+\sum_{k=1}^{n}\frac{u^{(k)}(\lambda)}{k!}(a_{k-1}-\gamma_{k-1}), \end{aligned} \tag{16}$$

where

$$\begin{aligned} J_c &= \int_{\lambda-l}^{\lambda+l}\frac{coswz}{z-\lambda}dz = -2\text{sin}\,(w\lambda)\text{Si}\,(wl), \\ J_s &= \int_{\lambda-l}^{\lambda+l}\frac{sinwz}{z-\lambda}dz = 2\text{cos}\,(w\lambda)\text{Si}\,(wl), \\ a_{k-1} &= \int_{\lambda-l}^{\lambda+l}(z-\lambda)^{k-1}(e^{iwz})dz, \end{aligned}$$

and

$$\gamma_{k-1}=\int_{\lambda-l}^{\lambda+l}e^{iw\lambda}(z-\lambda)^{k-1}dz=\frac{e^{iw\lambda}l^k}{k}(1-(-1)^k).$$

Theorem 3.1. *If* $a_{k-1}=\int_{\lambda-l}^{\lambda+l}(z-\lambda)^{k-1}(e^{iwz})dz$ *and* $a_i=0$ *for* $i\in\mathbb{Z}, i<0$, *then*

$$iwa_{k-1}+(k-1)a_{k-2}=l^{k-1}e^{iw\lambda}[e^{iwl}-(-1)^{k-1}e^{-iwl}], \tag{17}$$

and the non-homogeneous linear recurrence relation holds $\forall k=1(1)n$.

Proof. Let

$$a_{k-1}=\int_{\lambda-l}^{\lambda+l}(z-\lambda)^{k-1}(e^{iwz})dz.$$

Applying the method of integration by parts, we have

$$\begin{aligned} a_{k-1} &= \left[(z-\lambda)^{k-1}\frac{e^{iwz}}{iw}\right]_{\lambda-l}^{\lambda+l}-\frac{k-1}{iw}\int_{\lambda-l}^{\lambda+l}(z-\lambda)^{k-2}(e^{iwz})dz \\ &= \frac{l^{k-1}e^{iw\lambda}}{iw}[e^{iwl}-(-1)^{k-1}e^{-iwl}]-\frac{k-1}{iw}a_{k-2} \\ &\implies iwa_{k-1}+(k-1)a_{k-2}=l^{k-1}e^{iw\lambda}[e^{iwl}-(-1)^{k-1}e^{-iwl}], \end{aligned}$$

which completes the proof of the theorem. □

The recurrence relation (17) can be rewritten as

$$a_k + B(k-1)a_{k-1} = l^k ABC, k = 0, 1, 2, ..$$

where

$$A = e^{iw\lambda}, B = \frac{1}{iw}, C = e^{iwl} - (-1)^{k-1}e^{-iwl}.$$

On solving by following standard method of solution of recurrence relation, its particular solution is obtained as

$$a_k = A\sum_{i=1}^{k-1}(-1)^i \frac{k!}{(k-i)!}(e^{iwl} - (-1)^{k-i}e^{-iwl})B^{i+1}l^{k-i} + k!(-B)^k a_0, \tag{18}$$

with $a_0 = \frac{2}{w}e^{iw\lambda}\sin wl$.

3.2 Error Analysis

On the domain Ω, suppose that the function $u(z)$ is infinitely differentiable. Let us denote the error associated with the scheme designed for the numerical integration of $J(u)$ by $E_J(u)$. Then,

$$\begin{aligned} |E_J(u)| &\leq |J_o(u) - R_o(u)| + e^{iw\lambda}|I(u) - T_n(u)| \\ &= |E_o(u)| + |E_I(u)|, \end{aligned} \tag{19}$$

where

$$E_o(u) = J_o(u) - R_o(u),$$

and

$$E_I(u) = e^{iw\lambda}|I(u) - T_n(u)|,$$

are the error terms associated with the quadrature rules $R_o(u)$ and $T_n(u)$ designed for the approximation of the Filon type oscillatory integral $J_o(u)$ and CPV integral $I(u)$ respectively. However,

$$\begin{aligned} |E_o(u)| &\leq \frac{M_{n+1}}{(n+1)!}\left|\int_{\lambda-l}^{\lambda+l}(z-\lambda)^{n+1}(e^{iwz} - e^{iw\lambda})dz\right| \\ &\leq \frac{M_{n+1}}{(n+1)!}\left[\left|\int_{\lambda-l}^{\lambda+l}(z-\lambda)^{n+1}e^{iwz}dz\right| + 2\frac{|l|^{n+2}}{n+2}\right], \end{aligned} \tag{20}$$

where $M_{n+1} = \max_{\xi\in L}|u^{(n+1)}(\xi)|$. Further,

$$\begin{aligned} \left|\int_{\lambda-l}^{\lambda+l}(z-\lambda)^{n+1}e^{iwz}dz\right| &= \frac{1}{|w|}\left[2|l|^{n+1} + (n+1)\left|\int_{\lambda-l}^{\lambda+l}(z-\lambda)^n e^{iwz}dz\right|\right] \\ &\leq \frac{|l|^{n+1}}{|w|}\left[2 + (n+1)\int_{-1}^{1}|t|^n dt\right] \\ &= 4\frac{|l|^{n+1}}{|w|}, \end{aligned}$$

where $z = \lambda + l$ and $-1 \leq t \leq 1$. As a result,

$$|E_o(u)| \leq \frac{2|l|^{n+1} M_{n+1}}{(n+1)!} \left[\frac{2}{|w|} + \frac{|l|}{n+2}\right].$$

Since, from equation(2.14) it is evident that

$$|E_I(u)| \leq \frac{2|l|^{11} C M_{11}}{(11)!}; \;\; 0 < C < 1,$$

we get

$$|E_J(u)| \leq \frac{2|l|^{11} M_{11}}{(11)!} \left[C + \frac{2}{|w|} + \frac{|l|}{12}\right], \tag{21}$$

for $n = 10$. That is, if the Taylor's series is truncated after first eleven terms then the scheme will provide at least 10 decimal place of accuracy for an integral of the type(1.2). This fact is vividly seen when the proposed scheme is applied for the numerical approximation of such type of integrals.

4 Approximation of Integrals of the Type $H(u)$

$$H(u) = H\int_{\lambda-l}^{\lambda+l} \frac{u(z)}{(z-\lambda)^n} dz; n \in \mathbb{N} - \{1\}, \tag{22}$$

where the function f is differentiable a sufficient number of times in

$$\Omega = \{z \in \mathbb{C} : |z - \lambda| < \rho = r|l|, r > 1\},$$

of the complex plane $\mathbb{C}$ and L joining the points $\lambda - l$ to $\lambda + l$ lying in the disc Ω.

It is seen that rules designed for the integration of the integral (1.1) numerically lead to uncontrolled instability when these are used to approximate the integral provided in equation (22). This is due to the presence of singular point λ of order $\alpha > 1$ on the path of integration L. The integral defined in equation (22) is called hyper singular integral in complex plane.

A substantial work due to the eminent researchers exists in literature for the numerical integration of its real counter part

$$J^* = H\int_a^b \frac{u(x)}{(x-c)^2} dx; a < c < b. \tag{23}$$

The author in [4] proposed a new algorithm whereas authors in [7] generated Gauss type quadrature rules for Cauchy principal value and Hadamard finite part integrals. However, very few rules are available in the literature of numerical integration for the former.

Therefore, here we've presented a numerical approach constructed with the help of the rules designed for the integration of (1.1) numerically for the approximation of the integral (22).

4.1 Scheme for Numerical Evaluation of Complex Hyper Singular Integral

Since $u(z)$ is analytic and infinitely differentiable on the disc Ω, expanding $u(z)$ about the point $z = \lambda$, we obtain

$$\begin{aligned}\frac{u(z) - \sum_{k=0}^{n-2} c_k (z-\lambda)^k}{(z-\lambda)^{n-1}} &= \sum_{j=0}^{\infty} c_{j+n-1}(z-\lambda)^j; \\ &= h(z); \text{(say)}\end{aligned}$$

where $c_k = \frac{u^{(k)}(\lambda)}{k!}$ is the Taylor's coefficient and $j = k - n + 1$. As a result, the integral given in(22) reduces into

$$\begin{aligned}H(u) &= H\int_L \frac{u(z)}{(z-\lambda)^\alpha}dz \\ &= P\int_L \frac{h(z)}{z-\lambda}dz + \sum_{k=0}^{n-2}\int_L \frac{c_k}{(z-\lambda)^{n-k}} \\ &= I_h + \sum_{k=0}^{n-2} I_{nk},\end{aligned} \tag{24}$$

where

$$I_h = P\int_L \frac{h(z)}{z-\lambda}dz, \tag{25}$$

and

$$I_{nk} = \int_L \frac{c_k}{(z-\lambda)^{n-k}}. \tag{26}$$

Further, since the function $h(z)$ is analytic in the domain Ω, the first integral appearing on the right hand side of the equation (24) is a Cauchy type singular integral. Therefore, the class of quadrature rules $T_k(u); k = 1, 2, 3$ as constructed in this paper may be applied for its numerical approximation.

However, the integral $I_{nk}(u)$ is analytically a diverging integral and diverges for $(n-k) > 1$. Moreover, it is a hyper singular integral and its finite (Hadamard finite part) can be evaluated by transforming the integral onto the real axis with the help of the transformation

$$z = \lambda + lt; -1 \le t \le 1.$$

Now, by using this transformation the integral given in equation (26) is reduced into

$$\begin{aligned} I_{nk} &= \int_{\lambda-l}^{\lambda+l} \frac{c_k}{(z-\lambda)^{n-k}} dz \\ &= \frac{c_k}{l^{n-k-1}} \int_{-1}^{1} \frac{dt}{t^{n-k}} \end{aligned} \tag{27}$$

$$= \begin{cases} \frac{2c_k}{l^{n-k-1}} \int_0^1 \frac{dt}{t^{n-k}} & \text{, for} \quad (n-k) \text{ even ,} \\ 0 & \text{, for} \quad (n-k) \text{ odd.} \end{cases}$$

The integral appears for $(n-k)$ as even i.e.

$$\int_0^1 \frac{dt}{t^{n-k}},$$

is a hyper singular integral on real axis. To evaluate this integral, we consider the convergent integral

$$\int_{\epsilon}^1 \frac{dt}{t^{n-k}}, \tag{28}$$

and its value is obtained as

$$\frac{1-\epsilon^{k-n+1}}{k-n+1} = \frac{1}{k-n+1} - \frac{1}{k-n+1}\frac{1}{\epsilon^{n-k-1}}.$$

Now, letting $\epsilon \to 0$; (Of course the limit does not exist and so, Hadamard suggested to simply ignore the unbounded contribution of $\underset{\epsilon\to 0}{lim}\frac{1}{\epsilon}$ (Ref. Kai Diethelm [3], pp.233) and to assign the value of remaining finite expression) the finite part value of our original integral

$$I_{nk} = \begin{cases} \frac{2c_k}{l^{n-k-1}(k-n+1)} & \text{, for} \quad (n-k) \text{ even ,} \\ 0 & \text{, for} \quad (n-k) \text{ odd.} \end{cases}$$

According to [6], "as far as equality is concerned, the common rules for ordinary integrals are also valid for finite part integrals, but rules concerning inequalities are not applicable".

Therefore, the integral

$$H(u) \approx T_h + \sum_{k=0}^{n-2} \frac{2c_k}{h^{n-k-1}(k-n+1)},$$

where T_h is the quadrature rule meant for the numerical integration of the complex CPV integral I_h.

To verify the accuracy of the proposed scheme numerically, the scheme is applied over some standard test integrals already considered by different researchers. The results of their numerical approximations are given in Table 1 and Table 2. The integral is evaluated by the proposed scheme with the help of the two parametric rule $T_3(u)$ meant for the numerical evaluation of the complex CPV integral.

Table 1: Numerical evaluation of Cauchy type oscillatory integral.

w	n	**Approximation of** $J=\int_{-1}^{1} e^{iwx}\frac{e^x}{x}dx$	**Absolute Error**
10	4	$-0.133437973630331 + 3.400402005492587i$	1.5×10^{-4}
	6	$-0.133526758349377 + 3.400524589927999i$	1.7×10^{-6}
	8	$-0.133527790899802 + 3.400525943643699i$	1.2×10^{-8}
	10	$-0.133527798946889 + 3.400525952073923i$	6.4×10^{-10}
20	4	$0.107976715820401 + 3.077254563155467i$	4.4×10^{-5}
	6	$0.107939131902981 + 3.077231429744144i$	4.0×10^{-7}
	8	$0.107938778934500 + 3.077231253498199i$	2.9×10^{-9}
	10	$0.107938776647233 + 3.077231253926289i$	6.1×10^{-10}
40	4	$0.043517042181267 + 3.192609908219088i$	7.4×10^{-5}
	6	$0.043466627164752 + 3.192662066140842i$	1.1×10^{-6}
	8	$0.043466096017442 + 3.192663030447849i$	1.2×10^{-8}
	10	$0.043466092145451 + 3.192663041465206i$	6.2×10^{-10}
80	4	$-0.029149137524665 + 3.145962386539963i$	6.2×10^{-5}
	6	$-0.029210774742709 + 3.145964054630281i$	6.9×10^{-7}
	8	$-0.029211455759037 + 3.145964068957362i$	5.8×10^{-9}
	10	$-0.029211460936310 + 3.145964068945753i$	6.2×10^{-10}
160	4	$0.003252121838603 + 3.160386347432382i$	5.9×10^{-5}
	6	$0.003195885399764 + 3.160403391201590i$	6.8×10^{-7}
	8	$0.003195278879239 + 3.160403696142851i$	6.2×10^{-9}
	10	$0.003195274373740 + 3.160403699536554i$	6.2×10^{-10}
320	4	$0.006315272598841 + 3.144016839268697i$	5.6×10^{-5}
	6	$0.006259634806611 + 3.144019130660324i$	6.0×10^{-7}
	8	$0.006259036999550 + 3.144019172305005i$	5.1×10^{-9}
	10	$0.006259032576892 + 3.144019172775769i$	6.2×10^{-10}

Table 2: Numerical evaluation of complex hyper singular integrals.

Rules	**Approximation of** $\mathbf{H_1 = H\int_{-i}^{i}\frac{e^z}{z^2}dz}$	**Absolute Error**	**Approximation of** $\mathbf{H_2 = H\int_{-i}^{i}\frac{e^z}{z^3}dz}$	**Absolute Error**
$T_1(u)$	$2.972770752411771i$	5.9×10^{-11}	$2.327856361038685i$	4.5×10^{-12}
$T_2(u)$	$2.972770752415014i$	5.6×10^{-11}	$2.327856361038938i$	4.3×10^{-12}
$T_3(u)$	$2.972770752421825i$	4.9×10^{-11}	$2.327856361039460i$	3.8×10^{-12}
Exact Value	$2.972770752470646i$		$2.327856361043219i$	

5 Conclusion

The numerical evaluation of Cauchy type integrals of oscillatory functions was investigated using an interpolatory type methodology combined with a quasi exact quadrature method. This strategy is simple to implement. Table 1 indicates that whatever be the value of w, quick convergence can be achieved. Further, both for a given number of points and increasing w, and for a growing number of points and fixed w, the accuracy of the proposed schemes improves.

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